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Ultimate Bounded-ness of Periodic Solutions of A Class of Liénard Type p-Laplacian Equation with Multiple Deviating Arguments

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Abstract:
In this article, we investigate quite formally the existence of solutions followed by investigating the ultimate bounded-ness of the following Liénard type p-Laplacian equation with multiple deviating arguments :

$$\left.\begin{array}{r}(\phi_p(x'))' + \mu f_1(x)(\phi_p(x'))^2 + f_2(x)(\phi_p(x')) \\ +\beta(t)\displaystyle\sum_{i=1}^{n} g(t, x(t-\tau_i(t))) \quad = e(t)\end{array}\right\} (L)$$

where $\mu > 0$, f_1, f_2, $e \in C(\mathbb{R},\mathbb{R})$, $\beta(t) > 0$, $\tau_i(t) > 0$ are two T-periodic functions with $\int_0^T e(t)dt = 0$, $T > 0$, $g(t,x)$is continuous and $g(t,.) = g(t+T,.)$ and for $p > 1$, $\phi_p : \mathbb{R} \to \mathbb{R}$, $((\phi_p(x')))' = \frac{d}{dt}\{|\frac{dx}{dt}|^{p-2}\frac{dx}{dt}\}$ is a one-dimensional p-Laplacian. The study employs a combination of the Manásevich-Mawhin continuation theorem in settling the question of existence of periodic solutions and the Lyapunov second method in providing a framework for obtaining bounded-ness of such solutions. An example illustrates our results.

Keywords: Ultimate boundedness, Liénard equation, Manásevich-Mawhin continuation theorem, $p-$ Laplacian

AMS(MOS) Subject classification : 34B15, 34C05, 34C25

1. Introduction

In the study of dynamical systems (which include differential equations), Liénard equation is a second order differential equation named after its proponent, the French physicist Alfred-Marie Liénard. It arose from the development of radio and vacuum tube technology. Liénard and modified (or Liénard-type) equation model oscillating circuits. Current studies which involve practical problems concerning mechanics, engineering technique fields, economy, control theory, physics, chemistry, biology, medicine, atomic energy, information theory are associated with Liénard or the modified equation.

Qualitative questions on stability, bounded-ness, convergence and existence of periodic solutions of the equation are linked in the literature with the contributions of the following mathematicians : Liénard [10], Krasovskii [9], Burton [1], Heidel [7], Hara and Yoneyama [5], Gao and Zhao [4], Cantarelli [2], Manásevich and Mawhin [12], Liu and Huang [11], Cheng and Ren [4], Huo and Wu [8], Cemil

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Tunç[14, and references therein].

In this article, the author established certain sufficient conditions under which periodic solutions of equation (L) exist and are ultimately bounded. Our motivation springs up from the recent results of C. Tunç [14 and references therein] and a host of other prolific works of ([3], [4], [8], [11]). The paper is structured into four sections. After the introduction, Section 2 is devoted to a review of fundamental concepts and results used in subsequent sections. In Section 3 main results are stated and established. In conclusion, an illustrative example is provided in Section 4.

2. Preliminary notes.

In order to provide a suitable framework for subsequent chapters, we begin by reviewing the following fundamental notions, concepts and lemmas:

Definitions

Consider a system of differential equations

$$x' = f(t,x) \tag{2.1}$$

where $f(t,x) \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$, we say that

(i) a solution $x(t,t_0,x_0)$ of (2.1) is bounded if $\exists$ an $\alpha > 0$ such that $|x(t,t_0,x_0)| < \alpha \ \forall \ t > t_0$ where α may depend on each solution,

(ii) the solutions of (2.1) are ultimately bounded if $\exists$ a $B > 0$ and a $T > 0$ such that for every solution $x(t,t_0,x_0)$ of (2.1), $|x(t,t_0,x_0)| < B \quad \forall t \geq t_0 + T$ where B is independent of the particular solution while T may depend on each solution.

(iii) the solutions of (2.1) are uniformly ultimately bounded for bound B if the T in (ii) is independent of t_0.

In what follows we define **A classical Liénard equation** as

$$x'' + f(x)x' + g(x) = 0 \tag{2.2}$$

where $f, g \in C^1(\mathbb{R})$, f even g odd. The corresponding non-homogeneous equation is

$$x'' + f(x)x' + g(x) = h(t) \tag{2.3}$$

where $h(t)$ is a continuous T-periodic function, $f(x)$, is positive and $g(x)$ is monotonically increasing.

The following are several examples of **Modified Liénard equations**

(i)

$$(\phi_p(x'))' + \mu f(x)\phi_p(x') + g(x) = 0 \tag{2.4}$$

where $p > 1$, $\mu > 0$, is a parameter and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

(ii) **Modified Liénard equation with one constant deviating argument**

$$x'' + \mu f(x)(x') + g(x(t-h)) = 0 \tag{2.5}$$

where $f, g, \tau : \mathbb{R} \to \mathbb{R}$ are continuous functions on $\mathbb{R}$ and $h > 0$ is a constant.

(iii) **Modified Liénard equation with a variable deviating argument**

$$x'' + f_1(x)(x') + f_2(x)(x') + g(x(t-\tau(t))) = e(t) \quad (2.6)$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions on $\mathbb{R}$ with $\tau(t)$ being a periodic function of period T and $h > 0$ is a constant.

(iv) **Modified Liénard equation with multiple variable deviating arguments**

$$(\phi_p(x'))' + \mu f_1(x)(\phi_p(x'))^2 + f_2(x)(\phi_p(x')) + \beta(t)\sum_{i=1}^{n} g(t, x(t-\tau_i(t))) = e(t) \quad L$$

where $\mu > 0$, f_1, f_2, $e \in C(\mathbb{R},\mathbb{R})$, $\beta(t) > 0$, $\tau_i(t) > 0$ are two T-periodic functions with $\int_0^T e(t)dt = 0$, $T > 0$ $g(t,x)$ is continuous and $g(t,.) = g(t+T,.)$ and for $p > 1$, $\phi_p : \mathbb{R} \to \mathbb{R}$, $((\phi_p(x'))' = \frac{d}{dt}\left\{|\frac{dx}{dt}|^{p-2}\frac{dx}{dt}\right\}$ is a one-dimensional p-Laplacian.

(iv) is the same as equation(L) which is of interest in this article.

Let q be the conjugate exponent of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Using the substitution $y = \phi_q(x')$ and the identity $\phi_q.\phi_p(s) = s$, (L) can be replaced by the following equivalent first order system :

$$\begin{aligned} x' &= \phi_q(y) \\ y' &= -\mu f_1(x)y^2 - f_2(x)y - g(t,x(t)) \\ &\quad -\beta(t)\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g'((x(s)y(s))ds \\ &\quad +e(t) \end{aligned} \quad (2.7)$$

In what follows we define

$$C_T^1 = \{x \in C^1(\mathbb{R}^2) : x(0) = x(T),\ x'(0) = x'\} \quad (2.8).$$

For $x \in C_T^1$, define

$$||x|| = |x|_\infty + |x'|_\infty \quad (2.9)$$

with

$$|x|_\infty = \max_{t\in[0,T]}|x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]}|x'(t)| \quad (2.10).$$

Then C_T^1 is a Banach space.

We denote throughout this paper

$$B_r = \{x \in C_T^1 : ||x|| \le r\} \quad (2.11)$$

and state the following salient assumptions :

(Σ_1) $\exists$ constants $d \ge 0$, $N \ge 0$ with $N(\frac{T}{2})^p < 1$ such that for $|x| > d$,

$$\langle x, \sum_{i=1}^{n} g(t, x(t-\tau_i(t)))\rangle \le NT|x|^p \quad (2.12)$$

and

$$\beta(t)\sum_{i=1}^{n} g(t, x(t-\tau_i(t))) - e(t) \ne 0 \quad (2.13)$$

(Σ_2) $\exists$ a sequence $\{r_i\}_{i=1}^{\infty}$, $r_i \in \mathbb{R}_+$, $r_i \to +\infty$, such that the Brouwer degree

$$\deg(G, B_{r_i} \cap \mathbb{R}^n, 0) \ne 0 \quad (2.14)$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$G(a) = \frac{1}{T}\int_0^T \beta(t)(e(t) - g(t,a))dt \quad (2.15)$$

We now introduce the following lemma which is used in proving one of our main results :

Lemma 2.1 (Manásevich-Mawhin see [12], [13])

Consider the equation

$$(\phi_p(x'(t))' = f(t, x, x') \qquad (2.16)$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(t, ., .) = f(t + T, ., .)$. Assume that

(Φ_1) For each $\lambda \in (0, 1)$, the equations

$$(\phi_p(x'(t))' = \lambda f(t, x, x') \qquad (2.17)$$

has no T-periodic solution on ∂B_r

(Φ_2) $G(a) = 0$ has no solution on $\partial B_r \cap \mathbb{R}^n$ where

$$G(a) = \frac{1}{T}\int_0^T f(t, a, 0)dt \qquad (2.18)$$

(Φ_3) The Brouwer degree

$$\deg(G, B_r \cap \mathbb{R}^n, 0) \neq 0 \qquad (2.19)$$

Then equation (2.16) has at least one T-periodic solution in $\overline{B}_r$.

3. Main Results.

Proposition 3.1

If the assumptions Σ_1 and Σ_2 hold, then equation (L) has at least one T-periodic solution.

Proof.

We employ Lemma 2.1 in the proof of Proposition 3.1 as follows :

Our first step is to show that the set of all possible T-periodic solutions of the homotopy equation

$$(\phi_p(x'))' + \lambda\mu f_1(x)(\phi_p(x'))^2 + \lambda f_2(x)(\phi_p(x') + \lambda\beta(t)\sum_{i=1}^{n} g(t, x(t - \tau_i(t))) = \lambda e(t), \ \lambda \in (0, 1) \qquad (3.1)$$

is a bounded subset of C_T^1. So suppose $x(t)$ is an arbitrary T-periodic solution of equation (3.1). Then integrating (3.1) from 0 to T and using the conditions $x(0) = x(T)$, $x'(0) = x'(T)$, we obtain

$$\int_0^T \beta(t)(\sum_{i=1}^{n} g(t, x(t - \tau_i(t))) - e(t) = 0 \qquad (3.2)$$

By continuity $\sum_{i=1}^{n}\int_0^T = \int_0^T\sum_{i=1}^{n}$ and moreover since $\beta(t) > 0$, (3.2) yields

$$\int_0^T (g(t, x(t - \tau_i)) - e(t) = 0. \qquad (3.3)$$

This means there is a $\eta \in [0, T]$ such that

$$(g(\eta, x(\eta - \tau_i(\eta)) - e(\eta)) = 0 \qquad (3.4)$$

for each $i = (1, 2, \cdots, n)$. We have thus reached a contradiction of (2.15). Hence

$$|x(\eta)| \leq d \qquad (3.5),$$

from whence we next have that

$$\begin{aligned} |x(t)| &= |x(\eta) + \int_{\eta}^{t} x'(\tau)d\tau| \\ &\leq d + \int_{\eta}^{t} x'(\tau)|d\tau, \\ & t \in [\eta, \eta + T] \end{aligned} \quad (3.6)$$

This leads to the following estimate :

$$\begin{aligned} |x(t)| = |x(t-T)| &= |x(\eta) - \int_{t-T}^{\eta} x'(\tau)d\tau| \\ &\leq d + \int_{t-T}^{\eta} |x'(\tau)|d\tau, \\ & t \in [\eta, \eta + T] \end{aligned} \quad (3.7)$$

It therefore follows that

$$\begin{aligned} |x|_\infty &= \max_{t\in[0,T]} |x(t)| \\ &= \max_{t\in[\eta,\eta+T]} |x(t)| \\ &\leq \max_{t\in[\eta,\eta+T]} \{d + \frac{1}{2}(\int_{\eta}^{t} |x'(\tau)|d\tau \\ & + \textstyle\int_{t-T}^{\eta} |x'(\tau)|d\tau)\} \\ &\leq d + \frac{1}{2}\int_0^T |x'(\tau)|d\tau \end{aligned} \quad (3.8)$$

We now choose two non-intersecting sets namely :

$$\begin{aligned} K_1 = \{t : t \in [0,T],\ |x(t)| > d\} \\ K_2 = \{t : t \in [0,T],\ |x(t)| \leq d\} \end{aligned} \quad (3.9)$$

Integrating from 0 to T the resulting expression from the multiplication of homotopy equation (3.1) by $x(t)$ we obtain

$$\int_0^T |x'(t)|^p dt = -\int_0^T \langle(\phi_p(x'(t)))', x(t)\rangle dt$$

$$\begin{aligned} . \quad &= \lambda\mu \int_0^T \langle f_1(x)(\phi_p(x'))^2, x(t)\rangle dt \\ & +\lambda \int_0^T \langle f_2(x)(\phi_p(x'), x(t)\rangle dt \\ & +\lambda\beta(t) \sum_{i=1}^{n} \int_0^T \langle g(t, x(t-\tau_i(t))), x(t)\rangle dt \\ & -\lambda \int_0^T \langle e(t), x(t)\rangle dt \\ &= \lambda\beta(t) \sum_{i=1}^{n} \int_0^T \langle g(t, x(t-\tau_i(t))), x(t)\rangle dt \\ & -\lambda \int_0^T \langle e(t), x(t)\rangle dt \\ &= \lambda\beta(t) \sum_{i=1}^{n} \int_{K_1} \langle g(t, x(t-\tau_i(t))), x(t)\rangle dt \\ & +\lambda\beta(t) \sum_{i=1}^{n} \int_{K_2} \langle g(t, x(t-\tau_i(t))), x(t)\rangle dt \\ & -\lambda \int_0^T \langle e(t), x(t)\rangle dt \end{aligned} \quad (3.10)$$

The summands in (3.9) hold for each $i = (1, 2, \cdots, n)$.

Therefore we have that

$$\int_0^T |x'(t)|^p dt$$

$$\begin{aligned} &= \lambda\beta(t) \int_{K_1} \langle g(t, x(t-\tau(t))), x(t)\rangle dt \\ & +\lambda\beta(t) \int_{K_2} \langle g(t, x(t-\tau(t))), x(t)\rangle dt \\ & -\lambda \int_0^T \langle e(t), x(t)\rangle dt \\ &\leq \int_0^T NT|x|^p dt \\ & + \int_0^T \max_{t\in[0,T],\ x\leq d} |g(t, x(t-\tau))||x(t)|dt \\ & + \int_0^T |e(t)||x(t)|dt \leq NT|x|_\infty^p \\ & +AT|x|_\infty, \end{aligned} \quad (3.11)$$

with $A = \max\{|g(t, x(t-\tau))|, \quad t \in [0,T],\ |x(t)| \leq d\} + |e|$.

The next step involves establishing the following claims :

(i) $\exists$ a constant $N_1 > 0$ such that $|x|_\infty \leq N_1$, and

(ii) $\exists$ a constant $N_2 > 0$ such that $|x'|_\infty \leq N_2$.

Claim (i)

By (3.11), we have that $\exists$ $N_\ell > N$ with $N_\ell(\frac{T}{2})^p < 1$ such that for large $|x|_\infty$,

$$\int_0^T |x'(t)|dt \leq N_\ell T|x|_\infty^p \qquad (3.12)$$

Using Hölder's inequality we have

$$\begin{aligned}\int_0^T |x'(t)|dt \leq & \int_0^T (|x'(t)|^p dt)^{\frac{1}{p}} (\int_0^T 1dt)^{\frac{p-1}{p}} \\ & -T^{\frac{p-1}{p}} (\int_0^T |x'(t)|^p dt)^{\frac{1}{p}} \qquad (3.13)\end{aligned}$$

Therefore by (3.8), (3.12), (3.13) we get

$$\begin{aligned}|x|_\infty &\leq d + \tfrac{1}{2}T^{\frac{p-1}{p}} (\int_0^T |x'(t)|^p dt)^{\frac{1}{p}} \\ &\leq d + \tfrac{1}{2} + N_\ell^{\frac{1}{p}} |x|_\infty \qquad (3.14)\end{aligned}$$

Since $N_\ell(\frac{T}{2})^p < 1$, (3.14) $\Rightarrow$

$$|x|_\infty \leq d(1 - \frac{T}{2}N_\ell^{\frac{1}{p}})^{-1}. \qquad (3.15)$$

Therefore $\exists$ M_1 such that

$$|x|_\infty \leq N_1 \qquad (3.16)$$

Claim (ii)

Since $x(0) = x(T)$ $\exists$ $t_0 \in [0, T]$ such that $x'(t_0) = 0$. By $\phi_p(0) = 0$, we have

$$\begin{aligned}|x'|_\infty^{p-1} &= \max_{t\in[0.T]} |\phi_p(x'(t))| \\ &= \max_{t\in[t_0,t_0+T]} |\int_{t_0}^t (\phi_p(x'(s)))'ds| \\ &\leq \mu|\int_{t_0}^t f_1(x(s)(\phi_p(x'(s)))^2 ds| \\ &\quad + \int_{t_0}^t f_2(x(s))(\phi_p(x'(s))ds| \\ &\quad + \beta(s)\sum_{i=1}^n \int_{t_0}^t |g(s, x(s-\tau_i(s)))|ds \\ &\quad + \int_{t_0}^t |e(s)|ds \\ &\quad + \mu\int_0^T |f_1(x(t)(\phi_p(x'(t)))^2|ds \\ &\quad + \int_0^T f_2(x(t))(\phi_p(x'(t))|dt \\ &\quad + \beta(t)\sum_{i=1}^n \int_0^T |g(t, x(t-\tau_i(t)))|dt \\ &\quad + \int_0^T |e(t)|dt \\ &\leq QT + T\max_{t\in[0,T],x\in N_1} |g(t, x(t-\tau)| \\ &\quad + T|e|_\infty \qquad (3.17)\end{aligned}$$

where QT is a bound for all the integrals involving the $f_i's$, ($i = 1, 2$)) which holds for each $i \in I\!N$, μ, $\beta(t) > 0$ and

$$\int_0^T |f_1(x(t)(\phi_p(x'(t)))^2|dt + \int_0^T |f_1(x(t)(\phi_p(x'(t)))^2|dt \leq QT$$

It therefore follows that $\exists$ N_2 such that

$$|x'|_\infty \leq N_2 \qquad (3.18)$$

Combination of (3.16) and (3.18) yields

$$||x|| = |x|_\infty + |x'|_\infty \leq N_1 + N_2 \qquad (3.19)$$

from whence we have that the set of all T-periodic solutions of equation (3.1) is a

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bounded subset of C_T^1. The other two hypotheses of Lemma (2.1), namely that

$\deg(G, B_r \cap \mathbb{R}^n, 0) \neq 0$ and that $G(a) = 0$

has no solution on $\partial B_r \cap \mathbb{R}^n$ and are satisfied for $G(a) = \dfrac{1}{T}\displaystyle\int_0^T (e(t) - g(x(t-\tau(a))dt$ and $r > N_1 + N_2 + d + 1$. Hence equation (L) has at least one solution in B_r by the Manásevich-Mawhin theorem.

For our next result, we consider equation (L) as well as the associated assumptions on all the underlying functions and we employ the equivalent first order system in (2.9).

Proposition 3.2

In addition to the fundamental assumptions earlier imposed on functions $f_1,\ f_2,\ g,\ e,\ \mu,\ \beta$, it is assumed that $\exists$ positive constants $C_1,\ C_2$ such that the following conditions hold :

(Π_1) $\displaystyle\sum_{i=1}^n g(t, x(t-\tau_i(t)) - e(t) > 0\ \forall\ t \in \mathbb{R},\ \tau_i > 0,\ |\sup_{i\in\mathbb{N},\ t\in[0,T]} x(t-\tau_i(t))| > C_1$

(Π_2) $\lim_{x\to-\infty} \sup_{t\in[0,T]} \dfrac{|\sum_{i=1}^n g(t, x(t-\tau_i(t))|}{|x|^{p-1}} \leq C_2$

(Π_3) $f_1(x)y^2 + f_2(x)y \geq \delta,\ f_i(x)g(t,x) > 0, (i = 1.2)\ \forall |x| \geq h,\ |y| \geq k,\ |e(t)| \leq m,\ \forall\, t \geq 0.$

(Π_4) $|\mu f_1(x)y^2 + f_2(x)y| \leq F\ \forall\ f_i(x) sgnx > 0,\ |x| \geq h$

(Π_5) $\phi_q(y) sgny > 0,\ |y| > k$ and $|\phi_q(y)| \to \infty$ as $y \to \infty$

then solutions of equation (L) are uniformly ultimately bounded. Next we state a preliminary result we shall employ in the proof of Proposition 3.2.

Lemma 3.3 (see [15, p.72] for proof)
Assume there exists a Liapunov function $V(t,x,y)$ defined on $0 < t < \infty,\ |x|, \infty,\ |y| \geq K > 0$ which satisfies the following conditions :

(i) $a(|y|) \leq V(t,x,y) \leq b(|y|)$, where $a(r)$ and $b(r)$ are continuous, increasing and $a(r) \to \infty$ as $r \to \infty$,

(ii) $\dot{V}_L(t,x,y) \leq -c(|y|)$ where $c(r) > 0$ is continuous.

Suppose that corresponding to each M there exists a Liapunov function $W(t,x,y)$ defined on $0 \leq t < \infty,\ |x| \geq K_1(M),\ |y| \leq M$ which satifies the following conditions :

(iii) $a_1(|x|) \leq W(t,x,y) \leq b_1(|x|)$, where $a_1(r)$ and $b_1(r)$ are continuous and $a_1(r) \to \infty$ as $r \to \infty$,

(iv) $\dot{W}_L(t,x,y) \leq 0$

Moreover, assume that choosing B such that $b(K) < a(B)$, there exists a Liapunov function $U(t,x,y)$ defined on $T \leq t < \infty$, $|x| < K_2 > 0$, $|y| \leq B$ which satisfies the following conditions :

(v) $a_2(|x|) \leq U(t,x,y) \leq b_2(|x|)$, where $a_2(r)$ and $b_2(r)$ are continuous and increasing $a_1(r) \to \infty$ as $r \to \infty$,

(vi) $\dot{U}_L(t,x,y) \leq -c_2(|x)$ where $c_2(r) > 0$ is continuous,

then solutions of equation (L) are uniformly ultimately bounded.

Proof of Proposition 3.2

Consider equation (L) whose equivalent system is

$$\begin{aligned} x' &= \phi_q(y) \\ y' &= -\mu f_1(x)y^2 - f_2(x)y - g(t,x(t)) \\ &- \beta(t)\sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t} g'((x(s)y(s))ds + e(t) \end{aligned} \tag{3.3.1}$$

with all the afore-mentioned assumptions, we establish that the solutions are uniformly ultimately bounded. Suppose $K > 0$ is a constant such that

$$K \geq 1 + m + \frac{\alpha}{F} + F(1+\beta) + k \tag{3.3.2}$$

where $\alpha = \max\{\beta|y|m + F|\phi_q(y)| - \beta y\phi_q(y)\} \geq 0$ with

$$|\phi_q(y)| \geq m + \frac{F}{\beta}(1 + \beta F + m) \tag{3.3.3}$$

for $|y| \geq K$. Choose a domain $0 \leq t < \infty$, $\max(|y| - K, |w| - K) \geq 0$, on which a Liapunov function

$$V(y,w) = p(y,w) + q(y,w) \tag{3.3.4}$$

is defined, with $p(y,w) = \frac{1}{2}(\beta y^2 + w^2)$ and

$$q(y,w) = \begin{cases} -Fysgnw & \text{for} \quad |y| \leq w \\ -Fwsgny & \text{for} \quad |y| \geq |w| \end{cases}$$

It can be observed that $V(y,w)$ is clearly continuous, positive and $V(y,w) \to \infty$ as $y^2 + w^2 \to \infty$ since

$$\begin{aligned} \frac{1}{2}(\beta y^2 + w^2) - F|y| &\geq \frac{1}{2}\beta y^2 \frac{1}{2}|w|(|w| - F) \\ &> \frac{1}{2}\beta y^2 \quad for|y| \leq |w|, \\ \frac{1}{2}(\beta y^2 + w^2) - F|w| &\geq \frac{1}{2}w^2\frac{1}{2}|y|(\beta|y| - F) \\ &> \frac{1}{2}w^2 \quad for|y| \leq |y|, \end{aligned}$$

from whence it follows that there are two functions $a(r)$ and $b(r)$ such that

$$\beta(|y| + |w|) \leq V(y,w) \leq b(|y| + |w|) \tag{3.3.7}$$

Next we have that

$$\begin{aligned}\dot{p}_{(L)}(y,w) &\leq -ay\phi_q(y)+\beta|y|m+|w|F \\ &\text{and} \\ \dot{q}_{(L)}(y,w) &= -F|w|+F\phi_q(y)sgnw \\ &\quad -F|e(t)|sgnw \qquad (3.3.8)\end{aligned}$$

for $|y|\leq|w|$ and

$$\begin{aligned}\dot{q}_{(L)}(y,w) &= Ff(x)sgny+F\beta|y| \\ &\leq F^2+F\beta|y| \qquad (3.3.9)\end{aligned}$$

for $|y|\geq|w|$. Hence for $|y|\geq|w|$, we have that

$$\begin{aligned}&\dot{V}_{(L)}(y,w) \\ &\leq -|y|(\beta|\phi_q(y)|-\beta m-F-F\beta-F^2) \\ &< 0 \qquad (3.3.10)\end{aligned}$$

since$|y|\geq K$, and for $|y|\leq|w|$, $|y|\geq K$

$$\begin{aligned}\dot{V}_{(L)}(y,w) &\leq -\frac{\beta}{2}|y|[|\phi_q(y) \\ &- |m]-\frac{1}{2}|\phi_q(y)|[\beta|y|F) \\ &- F(y|-m)<0 \qquad (3.3.11)\end{aligned}$$

and for $|y|\leq|w|$, $|y|\geq K$

$$\dot{V}_{(L)}(y,w)\leq Fm+\beta-F|w|<0 \quad (3.3.12)$$

(since $|z|\geq K$) Next choose a function

$$W(x,w)=\frac{\beta}{2}(x+\frac{w}{\beta})^2$$

on $|w|\leq M$ and $|x|\geq\max(h,\dfrac{M}{\beta})$. It is then clear that

$$\dot{W}_{(L)}(x,w)=-(x+\frac{w}{\beta})g(t,x)<0 \qquad (3.3.13)$$

With a choice of B such that $\beta(B)<a(B)$, a function

$$U(x,w)=\frac{\beta}{2}(x+\frac{w}{\beta})^2$$

is defined on the domain $|w|\leq B$, $|x|\geq\max(h,\dfrac{B}{\beta})$. This yields

$$\frac{\beta}{4}x^2\leq U(x,z)\leq\frac{3\beta}{4}x^2 \qquad (3.3.14)$$

and

$$\dot{U}_{(L)}(x,w)\leq-\frac{1}{2}xg(t,x)<0 \qquad (3.3.15)$$

Thus appealing to Lemma 3.3, we have that the solutions of equation (L) are uniformly ultimately bounded.

4. An Example.

Consider the equation

$$\begin{aligned}(\phi_p(x'))'\exp\pi(t)(\phi_p(x'))^2 \\ +\exp 2\pi(t)(\phi_p(x') \\ \exp(t)\sum_{i=1}^{n}g(t,x(t-\sin n\pi(t))) &= \frac{1}{2\pi}\cos t \quad (4.1)\end{aligned}$$

where $p=3$, $T=2\pi$, $\tau_i(t)=\sin n\pi(t)$

$$g_n(t,x)=\begin{cases} x^{2n}\exp\cos^n t+\dfrac{1}{n\pi}\cos t, & x\geq 0 \\ \dfrac{x^{2n}}{n!}\exp\pi^2\cos^n t & x<0 \\ (n=1,2,\cdots,) \end{cases}$$

By (4.1), we can obtain $C_1=\dfrac{1}{1000}$, $C_2=\dfrac{1}{n\pi^2}$, with $N(\dfrac{T}{2})^p<1$, $C_2=\dfrac{1}{108\pi^4}$. All the assumptions $\Pi_1-\Pi_5$ hold. Hence according to Proposition 3.2 and Lemma

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3.3, (4.1) has at least one 2π-periodic solution which is uniformly ultimately bounded.

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