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Asymptotic Behaviour of The Heat Equation as The Solution of The Black-Scholes Model

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Abstract We carry out the reduction of the Black - Scholes equation via series of transformations to obtain the heat equation by reversing the direction of time,so that the pay-off of the Black-Scholes become the initial value condition of the heat equation.The solution of the obtained heat equation is generalized as the solution of the Black-Scholes equation.Here we examine the 1-dimensional case and its extension to the multi-dimensional case.

Keywords:Black-Scholes Equation,Heat Equation,Ito's Formula.
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Introduction

The Black-Scholes equation has a passing similarity to the more common heat equation.The solution to the Black -Scholes equation subject to a general payoff function $C(S_T, T) = f(S_T)$ at the expiry date T for a general European payoff assuming that the drift Volatility and interest rate are both constant,can be obtained [1].By a series of transformations,the Black-Scholes equation can be reduced to the heat equation which implies that the derivative price can be obtained by first solving the equation with initial condition $A(x, 0) = f(x)$.Once the Black -Scholes equation has been transformed to the heat equation,then the solution of the heat equation becomes the solution of the Black -Scholes equation .In [7],the analytical solution of the fractional Black-Scholes Equation is calculated using the Laplace transform.[6] provide a solution to the price of an Option on a dividend paying equity with the aid of general Fourier transformation when the parameters in the Black-Scholes PDE are time dependent.We consider a case of multiple underlying assets.

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The Model

We consider assets paying a known dividend rate q_i for each asset i which possess a SDE [4]

$$ds_i(t) = (r - q_i)S_i(t)dt + \sum_{i,j=1}^{n} \sigma_{i,j} S_i(t) dW_j(t) \tag{1}$$

Let $V \in C^{2,1}(\mathbb{R}^n X[0, T])$ be a continuous function with continuous partial derivatives,then we obtain the Ito's-formula

$$\begin{aligned} dV = (&\frac{\partial v}{\partial t} + \sum_{i=1}^{n}(r - q_i)S_i \frac{\partial v}{\partial S_i} \\ &+ \frac{1}{2}\sum_{j,k=1}^{n} \sigma_{i,k}\sigma_{j,k} S_i S_j \frac{\partial^2 V}{\partial S_i S_j})dt + \\ &\sum_{j=1}^{n} \sigma_{i,j} S_i \frac{\partial V}{\partial S_i} dW_j(t) \end{aligned} \tag{2}$$

If we form a portfolio Π consisting of one option V and Δ_i of the underlying assets S_i by shorting the contingent claim (option) V and long Δ_i unit of the underlying assets S_i,we obtain the Multi-dimensional Black-Scholes formula for asset paying cer-

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tain known dividend.

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}C_{i,j}S_iS_j\frac{\partial^2 V}{\partial S_iS_j}+\sum_{i=1}^{n}(r-q_i)S_i\frac{\partial v}{\partial S_i}-rV=0 \quad (3)$$

with terminal condition
$V(S_1,...S_n,T)=P(S_1,...,S_n)$, $0\leq S_i\leq\infty$
We attempt to derive the heat equation from the Black-Scholes by series of transformation.
Let $x_i=ln(s_i)$, then $\frac{\partial x_i}{\partial S_i}=\frac{1}{S_i}$ and
$\frac{\partial v}{\partial S_i}=\frac{\partial v}{\partial x_i}.\frac{\partial x_i}{\partial S_i}=\frac{1}{S_i}\frac{\partial v}{\partial x_i}$
$\Longrightarrow \frac{\partial v}{\partial x_i}=S_i\frac{\partial v}{\partial S_i}$
Also,

$$\frac{\partial^2 v}{\partial S_i\partial S_j}=-\frac{1}{S_iS_j}\frac{\partial v}{\partial x_i}+\frac{1}{S_iS_j}\frac{\partial^2 v}{\partial x_i\partial x_j}$$

Substitute these into the equation (3),we have

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}C_{i,j}S_iS_j[-\frac{1}{S_iS_j}\frac{\partial v}{\partial x_i}+\frac{1}{S_iS_j}\frac{\partial^2 v}{\partial x_i\partial x_j}]+\sum_{i=1}^{n}(r-q_i)\frac{\partial v}{\partial x_i}-rV=0 \quad (4)$$

Re-arrange and simplify,we have

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{n}C_{i,j}\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_{i=1}^{n}(r-q_i-\frac{1}{2}C_{i,j})\frac{\partial v}{\partial x_i}-rV=0 \quad (5)$$

This expression is equivalent to

$$\frac{\partial v}{\partial t}+\frac{1}{2}D_s^TCD_sV+bD_sV-rV=0= \quad (6)$$

Where,

$$D_s=\begin{bmatrix}\frac{\partial}{\partial x_1}\\ .\\ .\\ .\\ \frac{\partial}{\partial x_n}\end{bmatrix},b=\begin{bmatrix}r-q_1-\frac{1}{2}C_{1,j}\\ .\\ .\\ .\\ r-q_n-\frac{1}{2}C_{n,j}\end{bmatrix}$$

C is a positive definite matrix.If there exist orthonormal matrix B such that

$$BCB^T=D=\begin{bmatrix}\lambda_1 & - & - & - & 0\\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & - & - & - & \lambda_n\end{bmatrix}\geq 0$$

where
$\lambda_1,---\lambda_n$ are the eigenvalues of C with corresponding eigenvector

$$\overrightarrow{\xi}=\begin{bmatrix}\xi_{i1}\\ .\\ .\\ .\\ \xi_{in}\end{bmatrix}.$$

If we use the change of variable $\overrightarrow{Z}=B\overrightarrow{s}$,so that
$D_s=B^TD_z$ and $D_s^T=BD_z^T$,then the equation (6)become

$$\frac{\partial v}{\partial t}+\frac{1}{2}BD_z^TCB^TD_zV+b^TB^TD_zV-rV=0 \quad (7)$$

$$\frac{\partial v}{\partial t}+\frac{1}{2}D_z^T(BCB^T)D_zV+(Bb)^TD_zV-rV=0 \quad (8)$$

with $V(Z_1---Z_{n,T}=P(Z_1---Z_n)$,the terminal condition,we have

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sum_{i=1}^{n}\lambda_i\frac{\partial^2 V}{\partial Z_i}+\sum_{i=1}^{n}(\xi^T\overrightarrow{b})\frac{\partial v}{\partial Z_i}-rV=0 \quad (9)$$

with terminal condition assumed to be the Dirac delta function.

Let $V = exp(W^T T + \beta(T-t))W$
Differentiating V with respect to t and Z_i,we obtain

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n}[\lambda_i w_i + (\xi^T \overrightarrow{b})]\frac{\partial w}{\partial Z_i} + \frac{1}{2}\sum_{i=1}^{n}\lambda_i \frac{\partial^2 w}{\partial Z_i} - [r + \beta - \frac{1}{2}(W^T DW) - \overrightarrow{b}^T B^T \overrightarrow{w}]w = 0 \quad (10)$$

If $w_i = -\frac{1}{\lambda_i}(\xi^T \overrightarrow{b})$ and
$\beta = -r + \frac{1}{2}(W^T DW) + \overrightarrow{b}^T B^T \overrightarrow{w}$,
then we obtain an equation without $\frac{\partial w}{\partial Z_i}$

$$\frac{\partial w}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\lambda_i \frac{\partial^2 w}{\partial Z_i} = 0 \quad (11)$$

which is a Heat equation.

Solution of The Heat Equation

Let W_t be a standard Brownian motion.If we consider a function $f(x+w_t)$,such that $f \in C^{2,1}(\mathbb{R}^n, [0,T])$, by Ito's formula

$$df(x+w_t) = \frac{\partial f(x+w_t}{\partial w_\tau} + \frac{1}{2}\frac{\partial^2 f(x+w_t)}{\partial w_\tau^2}d\tau \quad (12)$$

Here ,we assume x to be a parameter.If we integrate with respect to τ, we have

$$f(x+w_\tau) = f(x) + \int_0^\tau \frac{\partial f}{\partial w_s}dw_s + \frac{1}{2}\int_0^\tau \frac{\partial^2 f(x+w_s)}{\partial w_s^2}ds \quad (13)$$

where $f(x+w_\tau) = f(x)$ at $t=0 \Longrightarrow w_\tau = 0$
If we differentiate $f(x+w_\tau)$ with respect to x ,we have

$$df(x+w_t) = \frac{\partial f(x+w_\tau}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f(x+w_\tau)}{\partial x^2}dx \quad (14)$$

Comparing (12) and(14),we have

$$f(x+w_\tau) = f(x) + \int_0^\tau \frac{\partial f(x+w_s)}{\partial x}dw_s + \frac{1}{2}\int_0^\tau \frac{\partial^2 f(x+w_s)}{\partial x^2}ds \quad (15)$$

Taking expectation of both sides of (15),we have

$$E(f(x+w_\tau) = f(x) + \frac{1}{2}\int_0^\tau \frac{\partial^2 E[f(x+w_\tau)]}{\partial x^2}ds$$

Since $E(dw_s] = 0$.
Let $A(x,\tau) = E(f(x+w_\tau)$,then

$$A(x,\tau) = f(x) + \frac{1}{2}\int_0^\tau \frac{\partial^2 A(x,\tau)}{\partial x^2}ds \quad (16)$$

Differentiate (16) with respect to τ, we obtain

$$\frac{\partial A(x,\tau)}{\partial \tau} = \frac{1}{2}\frac{\partial^2 A(x,\tau)}{\partial x^2}$$

which is also the Heat equation [5].If we evaluate $A(x,\tau)$ at $\tau = 0$,we have

$$\begin{aligned} A(x,0) &= E[f(x+w_0) \\ &= E[f(x)] \\ &= f(x) \end{aligned}$$

Hence $A(x,\tau)$ satisfied the initial condition $A(x,0) = f(x)$.

Result

Let $\tau \in [0,T], A \in C^{2,1}(\mathbb{R}^n, [0,T))$.If $A(x,\tau)$ satisfied the heat equation (PDE) subject to the initial condition [2] $A(x,0) = f(x)$,then

$$\begin{aligned} A(x,\tau) &= E[f(x+\sigma w_\tau) \\ &= \frac{1}{\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} f(x+\sigma\xi)e^{-\frac{\xi^2}{2\tau}}d\xi \end{aligned}$$

Proof;
Let

$$A(x,\tau) = E[f(x+\sigma w_\tau)$$
$$= E[f(x) + \int_0^\tau \frac{\partial f(x+\sigma w_\tau)}{dx} dw_s$$
$$+ \frac{1}{2}\int_0^\tau \frac{\partial^2 f(x+\sigma w_\tau)}{\partial x^2} ds]$$
$$= f(x) + \frac{1}{2}\sigma^2 \int_0^\tau \frac{\partial^2 E[f(x+\sigma w_s)]}{\partial x^2} ds$$
$$\frac{\partial A(x,\tau)}{dt} = \frac{1}{2}\sigma^2 \frac{\partial^2 A(x,\tau)}{\partial x^2}$$

Next,we show that $A(x,\tau) = E[f(x+\sigma w_\tau)]$ satisfies the heat equation as follows

$$A(x,\tau) = E[f(x+\sigma w_\tau)]$$
$$= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(x+\sigma\xi) e^{\frac{-\xi^2}{2\tau}} d\xi$$

Let $x+\sigma\xi = \eta$,$d\xi = \frac{d\eta}{\sigma}$,then

$$A(x,\tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) e^{-\frac{(\frac{\eta-x}{\sigma})^2}{2\tau}} d\eta\sigma$$
$$= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$

$$\frac{\partial A}{\partial \tau} = -\frac{1}{2\sqrt{2\pi}\tau^{\frac{3}{2}}} \int_{-\infty}^{\infty} f(\eta) e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$
$$+ \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) \frac{(\eta-x)^2}{2\tau\sigma^2} e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$
$$= \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) \frac{(\eta-x)}{\sigma^2\tau} e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) \frac{-1}{\sigma^2\tau} e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$
$$+ \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} f(\eta) \frac{(\eta-x)^2}{\sigma^4\tau^2} e^{-\frac{(\eta-x)^2}{2\tau\sigma^2}} \frac{d\eta}{\sigma}$$

Comparing $\frac{\partial A}{\partial \tau}$ and $\frac{\partial^2 A}{\partial x^2}$,we observe that

$$\frac{\partial A}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 A}{\partial x^2}$$

Now, we could solve for the derivative price $V(s_1,...,s_n)$ subject to the terminal condition $V(s_1,...,s_n,T) = F(s_1,...,s_n)$ where $F(s_1,...,s_n)$ is a prescribed function that is the payoff function of the derivative.
$T - t = \tau$, if $\tau = 0$ then $T = t$ for the terminal payoff function of the derivative to be an initial condition for $A(x,t)$.If we follow through the various transformation,we observe that the relationship between $V(s_1,---,s_n,T)$ and $A(x_1...,x_n,\tau)$ is

$$V(s_1,..,s_n,T) = \alpha(s_1,...,s_n,\tau)$$
$$= \beta(s_1,...,s_n,\tau)e^{-r\tau}$$
$$= \gamma(ln((s_1,...,s_n),\tau)e^{-r\tau})$$
$$= A(ln(s_1,...,s_n)$$
$$+ (r - \frac{1}{2}\sum_{j=1}^{n} C_{i,j})\tau,\tau)e^{-r\tau}$$

In particular,the derivative payoff function can be written as

$$F(s_1,..,s_n) = V(s_1,...,s_n,T)$$
$$= A(ln(s_1,..,s_n),0), \tau = 0, T = t$$

Hence ,the initial condition on $A(x_i,\tau)$ at $\tau = 0$ is ,
$A(x_1,...,x_n,0) = F(e^{x_i}), \quad i = 1,2....n$
For example,in the case of a call option on multiple assets,we have
$A(x_i,0) = \max(e^{x_i} - k, 0) \quad i = 1...n$ as the payoff.
For heat equation with initial conditions,
$A(x_i,0) = F(e^{x_i})$

$$A(x_i,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(e^{x_i} + \sigma\sqrt{i}\xi) e^{\frac{-1}{2\xi^2}} d\xi_i$$
$$= \frac{1}{\sqrt{2\pi}} \Pi_{i=1}^{n} \int_{-\infty}^{\infty} F(e^{x_i} + \sigma\sqrt{i}\xi) e^{\frac{-1}{2\xi^2}} d\xi_i$$

Using the value of $A(x_i)$,we can write the derivative price as

$$V(s_1,.,s_n,T) = A((In(s_i,..,s_n)+ r - \frac{1}{2}\sum_{j=1}^{n} C_{i,j})\tau,\tau)e^{-r\tau}$$
$$= \Pi_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[e(In(s_i,.,s_n) + (r - \frac{1}{2}\sum_{j=1}^{n} C_{ij})\tau + \sigma\sqrt{i}\xi)]e\frac{-1}{2\xi^2}d\xi_i$$
$$= \Pi_{i=1}^{n} \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(S_i e^{r\tau+\sigma\sqrt{\tau}\xi} - \frac{1}{2}\sum_{j=1}^{n} C_{i,j}\tau)e\frac{-1}{2\xi^2}d\xi_i$$

At $t = 0$,we obtain the initial price of the derivatives

$$V(s_1,..,s_n) = \Pi_{i=1}^{n} \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(S_i(0)e^{r\tau+\sigma\sqrt{\tau}\xi} - \frac{1}{2}\sum_{j=1}^{n} C_{i,j}T)e\frac{-1}{2\xi^2}d\xi_i$$

We observe that the present value of the derivative depends on the expiry date T, the initial asset price $S_i(0)$,the volatility $C_{i,j}$,the risk free interest rate r and the specification of the payoff function $F(S_i(T))$.The derivative price can also be written in terms of expectation

$V(s_1,...,s_n) = e^{-rT}\Pi_i^n E[F(S_i(T))$

$= e^{-r}E[(F(S_1(T)))E((S_2(T)))..E(F(S_n(T)))]$

where

$$S_i(T) = S_i(0)e^{-rT+\sigma\sqrt{T\xi_i}-\frac{1}{2}\sum_{j=1}^{n} C_{i,j}T}$$
$$= S_i(0)e^{-rT+\sigma W_i(T)-\frac{1}{2}\sum_{j=1}^{n} C_{i,j}T}$$

$W_i(T)$ is a random variable $\sim N(0,T)$ with respect to some new probability measure p called the risk-neutral measure [3].$S_i(T)$ is the value of the derivative at time T for each underlying asset.

$F(S_i(T))$ is the payoff function of the derivative with respect to each underlying asset.

Conclusion

The price of Call option and Put option at time $t = 0$ can be determined with our solution.For a call option that delivers S_T at time T ,that is $F(S_T) = S_T$ for instance,
$C_0 = e^{-rT}E[F(S_T)]$, with S_T define earlier
$C_0 = e^{-rT}E[S_T]$
$C_0 = S_0$
which is what we expect.

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