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Numerical Solution of First Order Delay Differential Equation Using a Newly Developed Mathematical Expression for Evaluation of Delay Term

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Abstract

This study aims to numerically solve several first-order delay differential equations (DDEs) for step numbers k = 2, 3, and 4 by employing the Extrapolated Block Backward Differentiation Formulae Method (EBBDFM). This is accomplished by using a recently developed mathematical formula to evaluate its delay term. The continuous form of each step number was used to generate discrete schemes, which were then constructed using a matrix inversion technique and a linear multistep collocation approach. Applying this suggested method yielded results demonstrating the accuracy and efficiency of the step number incorporated with an extrapolated future point, which outperformed the other existing methods at Lower Computational Processing Unit Time (LCPUT), particularly when compared to step numbers of K = 3 and 2.

Keywords: *First Order Delay Differential Equations, Block Method, Extrapolated Backward Differentiation Formulae, Delay Term.*

Mathematics Subject Classification 2020: *60H10; 91G60; 60H40; 60H25.*

1. Introduction

The extrapolated block backward differentiation formulae methods were introduced to modify the performance of the existing conventional BDF in terms of efficiency, accuracy, consistency, convergence and stability. In recent years, many rigorous numerical studies have been carried-out in obtaining the approximate solutions of delay differential equations which revealed its advantages in real life applications. Several studies by researchers, including [1, 2, 3, 4, 5, 6], have demonstrated the real-world uses of numerical approaches in solving delay differential equations. These researchers used interpolation techniques to evaluate the delay term in medicine, engineering, physics, and economics. However, they encountered obstacles in the process of obtaining accurate results. Equating the orders of the interpolating polynomials and using numerical methods—which are incredibly challenging to implement to find a numerical solution for any modeled system—are among the challenges these researchers face. When the interpolation method shifts throughout the numerical integration from the beginning function to earlier values and when the initial function does not totally cooperate with the rest of the

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modeled system, discontinuities are produced. Discontinuities can also occur when the beginning function is not correctly modeled. Regarding mathematics, DDEs are distinct from ODEs due to the development of DDEs comprises a series of previous values of dependent variables and derivatives. The development of ODEs, on the other hand, depends entirely on the values now being held by these quantities. Discovering discrete solutions to first-order delay differential equations (DDEs) of the type defined by [7] is the objective of this research project that we are working on.

$$P'(t) = q(t, p(t), p(t-\tau)), \text{for}\, t > t_0, \tau > 0$$
$$p(t) = \phi(t), \text{for}\, t \le t_0 \qquad (1)$$

where $\phi(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay term and $p(t-\tau)$ is the solution of the delay term.

The formula from [15] was utilized by researchers [8, 9, 10, 11, 12, 13, 14] to investigate the delay term of first-order delay differential equations. The researchers discovered that the formula is less precise, requires more computation time, and cannot be utilized to solve various DDEs. Different types of delay differential equations, such as stochastic, advanced, Riccati, partial, and stochastic partial delay differential equations, are included in these classes. It is necessary that a precise mathematical formulations need to be developed to handle these problems and the ones that scholars have run across while using the formula from [15] to assess the delay term.

2. Development of The Method

The k-step Linear Multistep Method (LMM) was created by [16]. We plan and describe the discrete schemes of the Extrapolated Block Backward Differentiation Formulae Method (EBBDFM) for steps 2, 3, and 4. Furthermore, we use the matrix inversion technique to do this.

For $k = 2$ of (EBBDFM)

$$p_{u+1} = \frac{23}{12} zq_{u+1} - \frac{4}{3} zq_{u+2} + \frac{5}{12} zq_{u+3} + p_u$$
$$p_{u+2} = -\frac{5}{23} p_u + \frac{28}{23} p_{u+1} + \frac{22}{23} zq_{u+2} - \frac{4}{23} zq_{u+3}$$
$$p_{u+3} = -\frac{4}{23} p_u + \frac{27}{23} p_{u+1} + \frac{36}{23} zq_{u+2} + \frac{6}{23} zq_{u+3} \quad (2)$$

For $k = 3$ of (EBBDFM)

$$p_{u+1} = -\frac{197}{120} zq_{u+1} - \frac{17}{40} zq_{u+3} + \frac{7}{60} zq_{u+4} - \frac{19}{40} p_u + \frac{59}{40} p_{u+2}$$
$$p_{u+2} = \frac{197}{165} zq_{u+2} - \frac{76}{165} zq_{u+3} + \frac{17}{165} zq_{u+4} - \frac{9}{55} p_u + \frac{64}{55} p_{u+1}$$
$$p_{u+3} = \frac{17}{197} p_u - \frac{99}{197} p_{u+1} + \frac{279}{197} p_{u+2} + \frac{150}{197} zq_{u+3} - \frac{18}{197} zq_{u+4}$$
$$p_{u+4} = \frac{9}{197} p_u - \frac{64}{197} p_{u+1} + \frac{252}{197} p_{u+2} + \frac{288}{197} zq_{u+3} + \frac{60}{197} zq_{u+4} \quad (3)$$

For $k = 4$ of (EBBDFM)

$$p_{u+1} = -\frac{2501}{2478} zq_{u+1} + \frac{184}{1239} zq_{u+4} - \frac{29}{826} zq_{u+5} - \frac{268}{1239} p_u + \frac{724}{413} p_{u+2} - \frac{95}{177} p_{u+3}$$
$$p_{u+2} = -\frac{2501}{336} zq_{u+2} - \frac{97}{112} zq_{u+4} + \frac{31}{168} zq_{u+5} + \frac{19}{42} p_u - \frac{475}{112} p_{u+1} + \frac{1609}{336} p_{u+3}$$
$$p_{u+3} = \frac{7503}{8018} zq_{u+3} - \frac{963}{4009} zq_{u+4} + \frac{333}{8018} zq_{u+5} + \frac{413}{8018} p_u - \frac{1467}{4009} p_{u+1} + \frac{10539}{8018} p_{u+2}$$
$$p_{u+4} = -\frac{111}{2501} p_u + \frac{728}{2501} p_{u+1} - \frac{2124}{2501} p_{u+2} + \frac{4008}{2501} p_{u+3} + \frac{1644}{2501} zq_{u+4} - \frac{144}{2501} zq_{u+5}$$
$$p_{u+5} = -\frac{24}{2501} p_u + \frac{225}{2501} p_{u+1} - \frac{1000}{2501} p_{u+2} + \frac{3300}{2501} p_{u+3} + \frac{3600}{2501} zq_{u+4} + \frac{780}{2501} zq_{u+5}$$
$$(4)$$

2.1 *Essential Characteristics of the Approach*

In this case, we follow the conditions given by [17] and [18] to compute the regions of absolute stability for equations (2), (3), and (4), as well as the orders, error constants, consistency, and zero stability.

2.1.1 Order and Error Constant

To find the order and error constants for equation (2), these steps were taken:

$c_0 = c_1 = c_2 = c_3 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$ but

$$c_4 = \begin{pmatrix} -\frac{3}{8} & \frac{17}{138} & \frac{3}{46} \end{pmatrix}^T$$

Therefore, (2) has order $n=3$ and error constants, $-\frac{3}{8}\frac{17}{138}\frac{3}{46}$

Applying the same step to (3), we obtained:

$$c_0=c_1=c_2=c_3=c_4=\begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}^T$$

but

$$c_5=\begin{pmatrix}-\frac{19}{150} & -\frac{413}{4950} & \frac{111}{1970} & \frac{12}{985}\end{pmatrix}^T$$

Therefore, (3) has order $n=4$ and error constants, $-\frac{19}{150}-\frac{413}{4950}\frac{111}{1970}\frac{12}{985}$

Implementing the same step to (4), we have:

$$c_0=c_1=\begin{pmatrix}0 & 0 & 0 & 0 & 0\end{pmatrix}^T \text{ but}$$

$$c_2=\begin{pmatrix}-\frac{5}{14} & -\frac{2065}{5703} & \frac{1665}{8018} & \frac{180}{4009} & \frac{85525}{8018}\end{pmatrix}^T$$

Therefore, (4) has order $n=1$ and error constants, $-\frac{5}{14}-\frac{2065}{5703}\frac{1665}{8018}\frac{180}{4009}\frac{85525}{8018}$

2.1.2 Consistency

Following the condition stated by [17], if the order is more than 1, i.e. $n\geq 1$, then the Linear Multistep Method is considered consistent. Our suggested method, EBBDFM, is consistent since its order, as evaluated using (2), (3), and (4), is higher than 1, i.e. $n\geq 1$.

2.1.3 Zero Stability Investigation

According to [18], EBBDFM is considered zero stable if and only if the initial characteristic polynomial has no roots $r_s, s=1,2,3,\dots,n$ that are either simple or distinct and have an expression as $E(r)=\det(rX_2^{(i)}-X_1^{(i)})$ higher than 1 which satisfies $|r_i|\leq 1$ and the roots $|r_i|$

The zero stability for (2) is analyzed as

follows:

$$\begin{pmatrix}1 & 0 & 0\\ -\frac{28}{23} & 1 & 0\\ -\frac{27}{23} & 0 & 1\end{pmatrix}\begin{pmatrix}p_{u+1}\\ p_{u+2}\\ p_{u+3}\end{pmatrix}=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & \frac{5}{23}\\ 0 & 0 & \frac{4}{23}\end{pmatrix}\begin{pmatrix}p_{u-2}\\ p_{u-1}\\ p_u\end{pmatrix}+z\begin{pmatrix}\frac{23}{12} & -\frac{4}{3} & \frac{5}{12}\\ 0 & \frac{22}{23} & -\frac{4}{23}\\ 0 & \frac{36}{23} & \frac{6}{23}\end{pmatrix}\begin{pmatrix}q_{u+1}\\ q_{u+2}\\ q_{u+3}\end{pmatrix}+z\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}q_{u-2}\\ q_{u-1}\\ q_u\end{pmatrix}$$

where

$$X_2^{(1)}=\begin{pmatrix}1 & 0 & 0\\ -\frac{28}{23} & 1 & 0\\ -\frac{27}{23} & 0 & 1\end{pmatrix}, X_1^{(1)}=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & \frac{5}{23}\\ 0 & 0 & \frac{4}{23}\end{pmatrix}$$ a

nd $V_2^{(1)}=\begin{pmatrix}\frac{23}{12} & -\frac{4}{3} & \frac{5}{12}\\ 0 & \frac{22}{23} & -\frac{4}{23}\\ 0 & \frac{36}{23} & \frac{6}{23}\end{pmatrix}$

The first characteristic polynomial is given by;

$$E(r)=\det\left(rX_2^{(1)}-X_1^{(1)}\right) = \left|rX_2^{(1)}-X_1^{(1)}\right|=0. \quad (5)$$

Now we have,

$$E(r)=\left|r\begin{pmatrix}1 & 0 & 0\\ -\frac{28}{23} & 1 & 0\\ -\frac{27}{23} & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & \frac{5}{23}\\ 0 & 0 & \frac{4}{23}\end{pmatrix}\right|=\left|\begin{pmatrix}r & 0 & 0\\ -\frac{28}{23}r & r & 0\\ -\frac{27}{23}r & 0 & r\end{pmatrix}-\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & \frac{5}{23}\\ 0 & 0 & \frac{4}{23}\end{pmatrix}\right|$$

$$\Rightarrow E(r)=\begin{pmatrix}r & 0 & 1\\ -\frac{28}{23}r & r & -\frac{5}{23}\\ -\frac{27}{23}r & 0 & r-\frac{4}{23}\end{pmatrix}$$

The following are obtained using Maple 18 software,

$E(r)=r^3+r^2 \Rightarrow r^3+r^2=0$

$\Rightarrow r_1=-1, r_2=0, r_3=0$ .Since

$|r_i|<1,\ i=1,2,3$, (2) is zero stable.

By the same procedure for (3)

$$\begin{pmatrix} 1 & -\frac{59}{40} & 0 & 0 \\ -\frac{64}{55} & 1 & 0 & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ \frac{64}{197} & -\frac{252}{197} & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{u+1} \\ p_{u+2} \\ p_{u+3} \\ p_{u+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{19}{40} \\ 0 & 0 & 0 & \frac{9}{55} \\ 0 & 0 & 0 & -\frac{17}{197} \\ 0 & 0 & 0 & -\frac{9}{197} \end{pmatrix} \begin{pmatrix} p_{u-3} \\ p_{u-2} \\ p_{u-1} \\ p_{u} \end{pmatrix}$$

$$+z \begin{pmatrix} -\frac{197}{120} & 0 & -\frac{17}{40} & \frac{7}{60} \\ 0 & \frac{197}{165} & -\frac{76}{165} & \frac{17}{165} \\ 0 & 0 & \frac{150}{197} & -\frac{18}{197} \\ 0 & 0 & \frac{288}{197} & \frac{60}{197} \end{pmatrix} \begin{pmatrix} q_{u+1} \\ q_{u+2} \\ q_{u+3} \\ q_{u+4} \end{pmatrix} + z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_{u-3} \\ q_{u-2} \\ q_{u-1} \\ q_{u} \end{pmatrix}$$

where $X_2^{(2)} = \begin{pmatrix} 1 & -\frac{59}{40} & 0 & 0 \\ -\frac{64}{55} & 1 & 0 & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ \frac{64}{197} & -\frac{252}{197} & 0 & 1 \end{pmatrix}, X_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \frac{19}{40} \\ 0 & 0 & 0 & \frac{9}{55} \\ 0 & 0 & 0 & -\frac{17}{197} \\ 0 & 0 & 0 & -\frac{9}{197} \end{pmatrix}$ and

$$V_2^{(2)} = \begin{pmatrix} -\frac{197}{120} & 0 & -\frac{17}{40} & \frac{7}{60} \\ 0 & \frac{197}{165} & -\frac{76}{165} & \frac{17}{165} \\ 0 & 0 & \frac{150}{197} & -\frac{18}{197} \\ 0 & 0 & \frac{288}{197} & \frac{60}{197} \end{pmatrix}$$

The first characteristic polynomial is presented as;

$$E(r) = \det\left(r X_2^{(2)} - X_1^{(2)}\right)$$
$$= \left| r X_2^{(2)} - X_1^{(2)} \right| = 0. \qquad (6)$$

Now we have,

$$E(r)=\left|r\begin{pmatrix}1 & -\frac{59}{40} & 0 & 0\\ -\frac{64}{55} & 1 & 0 & 0\\ \frac{99}{197} & -\frac{279}{197} & 1 & 0\\ \frac{64}{197} & -\frac{252}{197} & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & \frac{19}{40}\\ 0 & 0 & 0 & \frac{9}{55}\\ 0 & 0 & 0 & -\frac{17}{197}\\ 0 & 0 & 0 & -\frac{9}{197}\end{pmatrix}\right|=\left|\begin{pmatrix}r & -\frac{59}{40}r & 0 & 0\\ -\frac{64}{55}r & r & 0 & 0\\ \frac{99}{197}r & -\frac{279}{197}r & r & 0\\ \frac{64}{197}r & -\frac{252}{197}r & 0 & r\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & \frac{19}{40}\\ 0 & 0 & 0 & \frac{9}{55}\\ 0 & 0 & 0 & -\frac{17}{197}\\ 0 & 0 & 0 & -\frac{9}{197}\end{pmatrix}\right| \Rightarrow E(r)=\begin{pmatrix}r & -\frac{59}{40}r & 0 & -\frac{19}{40}\\ -\frac{64}{55}r & r & 0 & -\frac{9}{55}\\ \frac{99}{197}r & -\frac{279}{197}r & r & \frac{17}{197}\\ \frac{64}{197}r & -\frac{252}{197}r & 0 & r+\frac{9}{197}\end{pmatrix}$$

Adopting Maple 18 software, we have:

$$E(r)=-\frac{197}{275}r^4-\frac{197}{275}r^3 \Rightarrow -\frac{197}{275}r^4-\frac{197}{275}r^3=0$$

$\Rightarrow r_1=-1, r_2=0, r_3=0, r_4=0$. Following that $|r_i|<1,\ i=1,2,3,4$, (3) is zero stable.

Following the same procedure for (4)

$$\begin{pmatrix}1 & -\frac{724}{413} & \frac{95}{177} & 0 & 0\\ \frac{475}{112} & 1 & -\frac{1609}{336} & 0 & 0\\ \frac{1467}{4009} & -\frac{10539}{8018} & 1 & 0 & 0\\ -\frac{728}{2501} & \frac{2124}{2501} & -\frac{4008}{2501} & 1 & 0\\ -\frac{225}{2501} & \frac{1000}{2501} & -\frac{3300}{2501} & 0 & 1\end{pmatrix}\begin{pmatrix}p_{u+1}\\ p_{u+2}\\ p_{u+3}\\ p_{u+4}\\ p_{u+5}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0 & \frac{268}{1239}\\ 0 & 0 & 0 & 0 & -\frac{19}{42}\\ 0 & 0 & 0 & 0 & -\frac{413}{8018}\\ 0 & 0 & 0 & 0 & \frac{111}{2501}\\ 0 & 0 & 0 & 0 & \frac{24}{2501}\end{pmatrix}\begin{pmatrix}p_{u-4}\\ p_{u-3}\\ p_{u-2}\\ p_{u-1}\\ p_{u}\end{pmatrix}$$

$$+g\begin{pmatrix}-\frac{2501}{2478} & 0 & 0 & \frac{184}{1239} & -\frac{29}{826}\\ 0 & -\frac{2501}{336} & 0 & -\frac{97}{112} & \frac{31}{168}\\ 0 & 0 & \frac{7503}{8018} & -\frac{963}{4009} & \frac{333}{8018}\\ 0 & 0 & 0 & \frac{1644}{2501} & -\frac{144}{2501}\\ 0 & 0 & 0 & \frac{3600}{2501} & \frac{780}{2501}\end{pmatrix}\begin{pmatrix}q_{u+1}\\ q_{u+2}\\ q_{u+3}\\ q_{u+4}\\ q_{u+5}\end{pmatrix}+z\begin{pmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}q_{u-4}\\ q_{u-3}\\ q_{u-2}\\ q_{u-1}\\ q_{u}\end{pmatrix} \text{ where}$$

$$X_2^{(3)}=\begin{pmatrix} 1 & -\frac{724}{413} & \frac{95}{177} & 0 & 0 \\ \frac{475}{112} & 1 & -\frac{1609}{336} & 0 & 0 \\ \frac{1467}{4009} & -\frac{10539}{8018} & 1 & 0 & 0 \\ -\frac{728}{2501} & \frac{2124}{2501} & -\frac{4008}{2501} & 1 & 0 \\ -\frac{225}{2501} & \frac{1000}{2501} & -\frac{3300}{2501} & 0 & 1 \end{pmatrix}, X_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & 0 & \frac{268}{1239} \\ 0 & 0 & 0 & 0 & -\frac{19}{42} \\ 0 & 0 & 0 & 0 & -\frac{413}{8018} \\ 0 & 0 & 0 & 0 & \frac{111}{2501} \\ 0 & 0 & 0 & 0 & \frac{24}{2501} \end{pmatrix} \text{ and } X_2^{(3)}=\begin{pmatrix} -\frac{2501}{2478} & 0 & 0 & \frac{184}{1239} & -\frac{29}{826} \\ 0 & -\frac{2501}{336} & 0 & -\frac{97}{112} & \frac{31}{168} \\ 0 & 0 & \frac{7503}{8018} & -\frac{963}{4009} & \frac{333}{8018} \\ 0 & 0 & 0 & \frac{1644}{2501} & -\frac{144}{2501} \\ 0 & 0 & 0 & \frac{3600}{2501} & \frac{780}{2501} \end{pmatrix}$$

The first characteristic polynomial is presented as;

$$\begin{aligned} E(r) &= \det\left(r X_2^{(3)} - X_1^{(3)}\right) \\ &= \left|r X_2^{(3)} - X_1^{(3)}\right| = 0. \end{aligned} \qquad (7)$$

Now we have,

$$E(r)=\left| r\begin{pmatrix} 1 & -\frac{724}{413} & \frac{95}{177} & 0 & 0 \\ \frac{475}{112} & 1 & -\frac{1609}{336} & 0 & 0 \\ \frac{1467}{4009} & -\frac{10539}{8018} & 1 & 0 & 0 \\ -\frac{728}{2501} & \frac{2124}{2501} & -\frac{4008}{2501} & 1 & 0 \\ -\frac{225}{2501} & \frac{1000}{2501} & -\frac{3300}{2501} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{268}{1239} \\ 0 & 0 & 0 & 0 & -\frac{19}{42} \\ 0 & 0 & 0 & 0 & -\frac{413}{8018} \\ 0 & 0 & 0 & 0 & \frac{111}{2501} \\ 0 & 0 & 0 & 0 & \frac{24}{2501} \end{pmatrix} \right|$$

$$=\left| \begin{pmatrix} r & -\frac{724}{413}r & \frac{95}{177}r & 0 & 0 \\ \frac{475}{112}r & r & -\frac{1609}{336}r & 0 & 0 \\ \frac{1467}{4009}r & -\frac{10539}{8018}r & r & 0 & 0 \\ -\frac{728}{2501} & \frac{2124}{2501}r & -\frac{4008}{2501}r & r & 0 \\ -\frac{225}{2501} & \frac{1000}{2501}r & -\frac{3300}{2501}r & 0 & r \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{268}{1239} \\ 0 & 0 & 0 & 0 & -\frac{19}{42} \\ 0 & 0 & 0 & 0 & -\frac{413}{8018} \\ 0 & 0 & 0 & 0 & \frac{111}{2501} \\ 0 & 0 & 0 & 0 & \frac{24}{2501} \end{pmatrix} \right|$$

$$\Rightarrow E(r)=\begin{pmatrix} r & -\frac{724}{413}r & \frac{95}{177}r & 0 & -\frac{268}{1239} \\ \frac{475}{112}r & r & -\frac{1609}{336}r & 0 & \frac{19}{42} \\ \frac{1467}{4009}r & -\frac{10539}{8018}r & r & 0 & \frac{413}{8018} \\ -\frac{728}{2501} & \frac{2124}{2501}r & -\frac{4008}{2501}r & r & -\frac{111}{2501} \\ -\frac{225}{2501} & \frac{1000}{2501}r & -\frac{3300}{2501}r & 0 & r-\frac{24}{2501} \end{pmatrix}$$

Adopting Maple 18 software, we have:

$$E(r)=\frac{93825015}{46360076}r^5+\frac{93825015}{46360076}r^4 \Rightarrow \frac{93825015}{46360076}r^5+\frac{93825015}{46360076}r^4=0$$

$\Rightarrow r_1=-1, r_2=0, r_3=0, r_4=0, r_5=0$ .Since $|r_i|<1,\ i=1,2,3,4,5$, (7) is zero stable.

2.1.4 Convergence

Looking at the suggested technique, we could say it is convergent having seen that equations (2), (3), and (4) are not only consistent but stable at zero.

2.1.5 Region of Absolute Stability

Figures 1 through 6 illustrate the plotting of the G- and H- regions of absolute stability of (2), (3), and (4). These plots were created using Map 18 and the MATLAB program, as shown in the following figure; 1-6

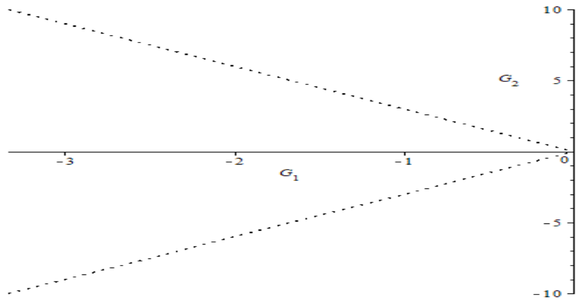

Fig.1. G -stability Region (EBBDFM) in (2)

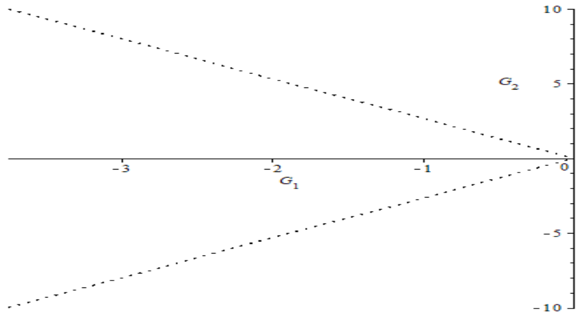

Fig.2. G -stability Region (EBBDFM) in (3)

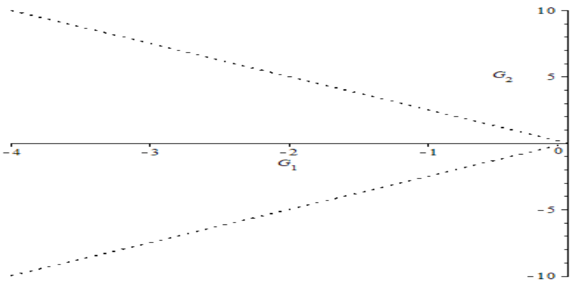

Fig.3. G -stability Region (EBBDFM) in (4)

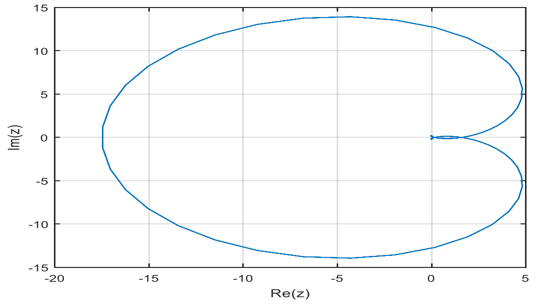

Fig.4. H -stability Region (EBBDFM) in (2)

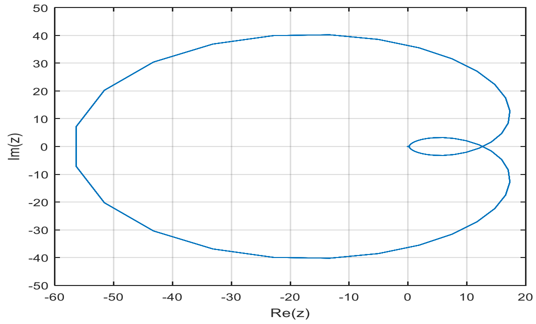

Fig.5. H -stability Region (EBBDFM) in (3)

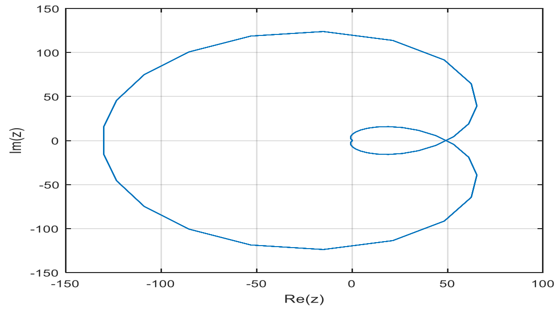

Fig.6. H -stability Region (EBBDFM) in (4)

Within the open-ended zone, lies the *G*-stability regions as shown in Figures 1-3, while within the enclosed H -stability region, are shown in Figures 4-6. Consequently, our proposed method's region of absolute stability is fulfilled.

3. EVALUATION OF THE DELAY TERM $(t-\tau)$

The newly developed mathematical expressions used in [19] to calculate and evaluate the delay term $(t-\tau)$ differ from those earlier researchers used as seen in online literature. It allows better and faster evaluations, calculations, performances, and gives accurate results. Before carrying-out the numerical experiments with a constant size of z = 0.01 to get numerical solutions of $P'(t)$ in Maple 18 software, the new mathematical expression for evaluation of the delay term shall be put into first-order DDEs with the (2), (3), and (4) of the proposed method.

1. Numerical implementation and Computation

Utilizing equations (2), (3), and (4) with a fixed step size of z = 0.01, this section aims to solve some first-order DDEs. To solve this, the Maple 18 software will be used to get the numerical solutions of $p'(t)$

Numerical Problems
Problem 1

$$p'(t) = -1000p(t) + p(t-(\ln(1000-1))),$$
$$0 \le t \le 3$$
$$p(t) = e^{-t}, t \le 0$$

Exact solution $p(t) = e^{-t}$

Problem 2

$$p'(t) = -1000p(t) + 997e^{-3}p(t-1) + (1000-997e^{-3}),$$
$$0 \le t \le 3$$
$$p(t) = 1 + e^{-3t}, t \le 0$$

Exact solution $p(t) = 1 + e^{-3t}$

We derived and applied the step numbers k = 2, 3 and 4 discrete schemes of the proposed method to obtain the most accurate numerical solutions of some first order delay differential equations. The step number of the proposed method with the Least Minimum Absolute Error at Lower Computational Processing Unit Time (LCPUT) of the two first order delay differential equations above indicates higher and better performance in terms of accuracy and efficiency than other step numbers of the method and other existing methods. The numerical solutions are presented in a tabular form in Table 1;

Table 1: Problem 1 Approximate Solutions

t	Exact Solution	k = 2 Approximate Solution	k =3 Approximate Solution	k= 4 Approximate Solution
0.1	0.990049834	0.990049834	0.990049835	0.990049835
0.2	0.980198673	0.980198674	0.980198674	0.980198674
0.3	0.970445534	0.970445536	0.970445534	0.970445534
0.4	0.960789439	0.96078944	0.960789439	0.96078944
0.5	0.951229425	0.951229425	0.951229425	0.951229426
0.6	0.941764534	0.941764532	0.941764534	0.941764534
0.7	0.93239382	0.932393821	0.93239382	0.93239382
0.8	0.923116346	0.923116347	0.923116348	0.923116347
0.9	0.913931185	0.913931183	0.913931186	0.913931185
1	0.904837418	0.904837419	0.904837418	0.90483742
1.1	0.895834135	0.895834135	0.895834136	0.895834136
1.2	0.886920437	0.886920434	0.886920437	0.886920437
1.3	0.878095431	0.878095431	0.878095432	0.878095431
1.4	0.869358235	0.869358236	0.869358235	0.869358236
1.5	0.860707976	0.860707976	0.860707977	0.860707977
1.6	0.852143789	0.85214379	0.852143789	0.852143789
1.7	0.843664817	0.843664817	0.843664817	0.843664817
1.8	0.835270211	0.835270211	0.835270211	0.835270212
1.9	0.826959134	0.826959135	0.826959134	0.826959134
2	0.818730753	0.818730754	0.818730754	0.818730753
2.1	0.810584246	0.810584245	0.810584247	0.810584246
2.2	0.802518798	0.802518798	0.802518798	0.802518798
2.3	0.794533603	0.794533603	0.794533603	0.794533603
2.4	0.786627861	0.786627862	0.786627861	0.786627861
2.5	0.778800783	0.778800784	0.778800783	0.778800783

2.6	0.771051586	0.771051586	0.771051586	0.771051586
2.7	0.763379494	0.763379493	0.763379495	0.763379495
2.8	0.755783741	0.755783742	0.755783741	0.755783742
2.9	0.748263568	0.748263568	0.748263569	0.748263568
3.0	0.740818221	0.740818219	0.740818221	0.74081822

CPUT of EBBDFM for k = 2 is 0.4s, k = 3 is 0.2 and k = 4 is 0.1s

Table 2: Problem 2 Approximate Solutions

t	Exact Solution	k = 2 Approximate Solution	k = 3 Approximate Solution	k = 4 Approximate Solution
0.1	1.970445534	1.970445542	1.970445533	1.970445533
0.2	1.941764534	1.941764528	1.941764534	1.941764534
0.3	1.913931185	1.913931195	1.913931187	1.913931185
0.4	1.886920437	1.886920444	1.886920432	1.886920436
0.5	1.860707976	1.86070797	1.860707978	1.860707984
0.6	1.835270211	1.835270223	1.83527021	1.835270211
0.7	1.810584246	1.810584253	1.810584246	1.810584246
0.8	1.786627861	1.786627856	1.786627862	1.78662786
0.9	1.763379494	1.763379504	1.763379495	1.763379494
1.0	1.740818221	1.740818227	1.74081822	1.740818224
1.1	1.718923733	1.718923728	1.718923733	1.718923733
1.2	1.697676326	1.697676335	1.697676325	1.697676326
1.3	1.677056874	1.677056881	1.677056874	1.677056874
1.4	1.65704682	1.657046816	1.657046819	1.65704682
1.5	1.637628152	1.637628158	1.637628152	1.637628153
1.6	1.618783392	1.618783397	1.618783393	1.61878339
1.7	1.600495579	1.600495575	1.600495579	1.600495579
1.8	1.582748252	1.582748258	1.582748252	1.582748252
1.9	1.565525439	1.565525443	1.565525439	1.565525438
2.0	1.548811636	1.548811632	1.548811637	1.548811638
2.1	1.532591801	1.532591809	1.532591802	1.532591801
2.2	1.516851334	1.51685134	1.516851334	1.516851335
2.3	1.501576069	1.501576066	1.501576069	1.501576068
2.4	1.486752256	1.48675226	1.486752255	1.486752256
2.5	1.472366553	1.472366557	1.472366552	1.472366555
2.6	1.458406011	1.458406009	1.458406011	1.458406012
2.7	1.444858066	1.44485807	1.444858067	1.444858066
2.8	1.431710523	1.431710528	1.431710523	1.431710524
2.9	1.418951549	1.418951547	1.418951549	1.41895155
3.0	1.40656966	1.406569664	1.40656966	1.406569662

CPUT of EBBDFM for k = 2 is 0.5s, k = 3 is 0.3 and k = 4 is 0.2s

5. Analysis of Results and Discussions

In this section, we analyzed the absolute errors between the exact and approximate solutions obtained after the numerical experiment using the proposed method. The results are presented in Table 3.

Table 3: Problem 1 Absolute Errors

t	k = 2 AbsoluteError	k = 3 AbsoluteError	k = 4 AbsoluteError
0.1	3.51832E-10	8.51832E-10	1.46081E-09
0.2	2.92245E-09	5.94245E-09	2.94246E-10
0.3	2.06149E-09	4.52492E-09	6.52493E-10
0.4	1.05768E-09	1.46677E-09	5.48678E-09
0.5	1.98286E-10	4.98286E-09	1.58928E-09
0.6	1.39425E-09	8.43487E-10	4.16752E-10
0.7	1.38405E-09	2.95052E-09	3.95053E-09
0.8	3.14364E-09	1.62336E-09	1.14365E-10
0.9	2.58123E-09	4.29772E-09	7.13282E-11
1	4.6504E-10	1.6504E-09	1.97405E-10
1.1	1.04472E-10	2.04472E-09	5.04471E-09
1.2	2.72716E-09	1.18158E-10	2.83842E-10
1.3	3.21561E-09	6.78439E-10	7.94386E-12
1.4	1.02194E-09	3.99806E-10	2.02193E-10
1.5	5.26058E-09	7.48422E-10	1.75941E-10
1.6	1.14379E-09	1.65211E-10	4.34788E-09
1.7	6.04616E-09	6.04616E-10	3.04615E-10
1.8	2.13272E-09	3.12272E-09	1.89727E-09
1.9	1.14664E-09	3.57638E-10	1.57637E-10
2	4.21018E-09	1.03202E-09	4.78981E-10
2.1	9.70187E-10	1.03981E-09	3.20812E-09
2.2	2.38522E-09	2.63478E-09	1.38521E-10
2.3	3.97666E-09	9.6766E-11	1.97665E-10
2.4	5.34447E-09	2.67553E-10	3.35465E-11
2.5	6.29595E-09	7.15049E-09	2.86952E-11
2.6	3.57626E-09	2.04566E-10	5.97433E-10
2.7	1.04685E-09	2.62147E-09	2.64146E-09
2.8	6.45275E-09	3.54725E-09	2.45275E-10
2.9	7.86653E-09	9.22435E-10	1.22434E-09
3	1.89172E-09	8.16179E-10	2.82717E-10

Table 4: Problem 2 Absolute Errors

t	k = 2 AbsoluteError	k = 3 AbsoluteError	k = 4 AbsoluteError
0.1	8.45149E-09	5.48508E-10	5.48508E-10
0.2	5.58425E-09	4.15751E-10	4.15751E-10
0.3	9.72877E-09	1.72877E-09	2.71228E-10
0.4	7.28284E-09	4.71716E-09	7.17157E-10
0.5	6.42506E-09	1.57494E-09	7.57494E-09
0.6	1.15887E-08	1.41127E-09	4.11272E-10
0.7	7.02981E-09	2.98128E-11	2.98128E-11
0.8	5.06655E-09	9.33447E-10	1.06655E-09
0.9	9.66315E-09	6.63147E-10	3.36853E-10
1	6.31828E-09	6.81718E-10	3.31828E-09
1.1	5.43193E-09	4.31926E-10	4.31926E-10
1.2	8.92897E-09	1.07103E-09	7.1031E-11
1.3	6.50184E-09	4.98165E-10	4.98165E-10
1.4	3.81506E-09	8.15057E-10	1.84943E-10
1.5	6.37823E-09	3.78227E-10	1.37823E-09
1.6	5.19386E-09	1.19386E-09	1.80614E-09
1.7	3.81227E-09	1.87734E-10	1.87734E-10

1.8	5.62601E-09	3.7399E-10	3.7399E-10
1.9	4.30046E-09	3.00463E-10	6.99537E-10
2	4.09403E-09	9.05974E-10	1.90597E-09
2.1	7.9931E-09	9.93103E-10	6.89737E-12
2.2	5.5083E-09	4.91699E-10	5.08301E-10
2.3	3.06606E-09	6.60556E-11	1.06606E-09
2.4	4.04003E-09	9.59972E-10	4.00282E-11
2.5	4.25899E-09	7.41015E-10	2.25899E-09
2.6	2.30522E-09	3.05224E-10	6.94776E-10
2.7	3.77706E-09	7.77059E-10	2.22941E-10
2.8	4.57092E-09	4.2908E-10	5.7092E-10
2.9	2.24764E-09	2.47639E-10	7.52361E-10
3	4.2594E-09	2.59401E-10	2.2594E-09

5.1 Graphical Presentation of the Evaluated Absolute Errors in Tables 3 and 4

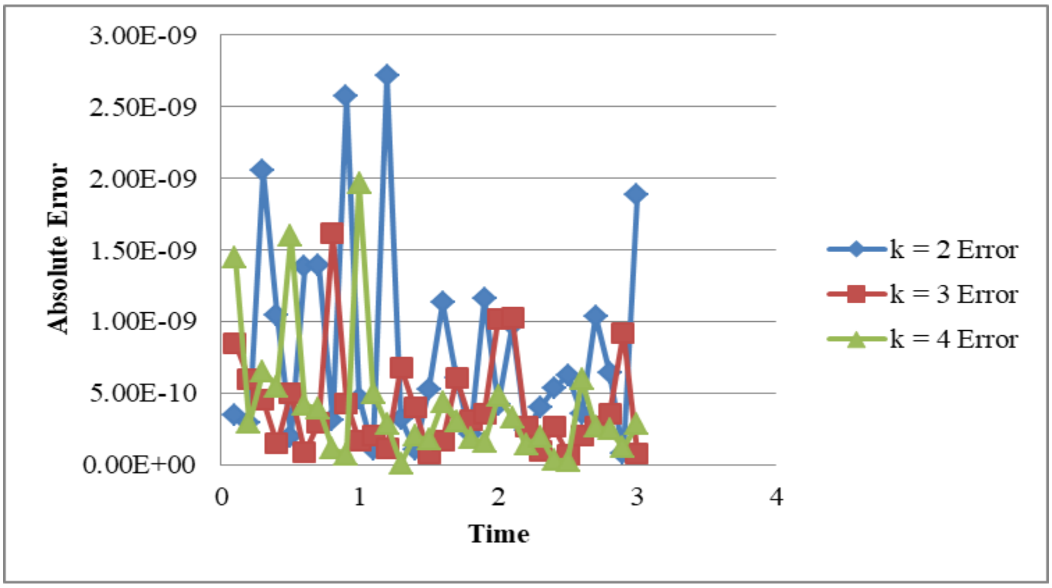

Fig. 7: The absolute error results for Example 1 are shown in Figure 7; they are plotted against time using EBBDFM (as shown by the colors) to illustrate the performance of the approach for step numbers k=2, 3, and 4 with varying Absolute Errors, the colorful lines indicate the performance.

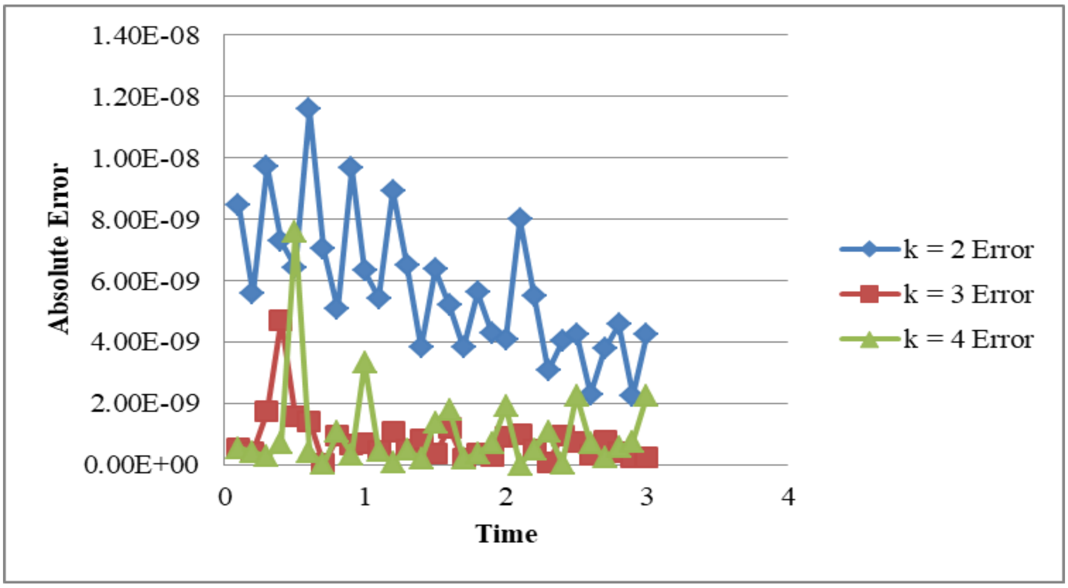

Fig. 8: The absolute error results for Example 2 are shown in Figure 8; they are plotted against time using EBBDFM (as shown by the colors) to illustrate the performance of the approach for step numbers k=2, 3, and 4 with varying Absolute Errors, the colorful lines indicate the performance.

5.2 Comparison of Results

To prove the advantage of the proposed method, we compared the results we obtained with other existing methods as shown in Tables 5 and 6.

EBBDFM = Extrapolated Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ and 4.

RBBDFM = Reformulated Block Backward

Differentiation Formulae Methods for step numbers $k = 3$ and 4 in [15].

CBBDFM = Conventional Block Backward Differentiation Formulae Method for step numbers $k = 2$ and 3 in [16].

MAEs = Minimum Absolute Errors. The compared results are presented in the table below;

Table 5: Comparison between the Minimum Absolute Errors of EBBDFM $k = 2, 3$ and 4 with [15, 16] for constant step size z = 0.01 of Problem 1

Numerical Method	**Compared MAEs with [15,16]**
EBBDFM MAE for k = 2	9.70E-10
EBBDFM MAE for k = 3	9.67E-11
EBBDFM MAE for k = 4	7.94E-12
RBBDFM MAE for k = 3	1.61E-07
RBBDFM MAE for k = 4	1.54E-08
CBBDFM MAE for k = 2	1.66E-05
CBBDFM MAE for k = 3	2.22E-07

Table 6: Comparison between the Minimum Absolute Errors of EBBDFM $k = 2, 3$ and 4 with [15, 16] for constant step size z = 0.01 of Problem 2

Numerical Method	**Compared MAEs with [15,16]**
EBBDFM MAE for k = 2	9.73E-09
EBBDFM MAE for k = 3	6.61E-11
EBBDFM MAE for k = 4	6.90E-12
RBBDFM MAE for k = 3	1.61E-07
RBBDFM MAE for k = 4	1.54E-08
CBBDFM MAE for k = 2	1.66E-05
CBBDFM MAE for k = 3	2.22E-07

5.3Graphical Presentation of Compared Results

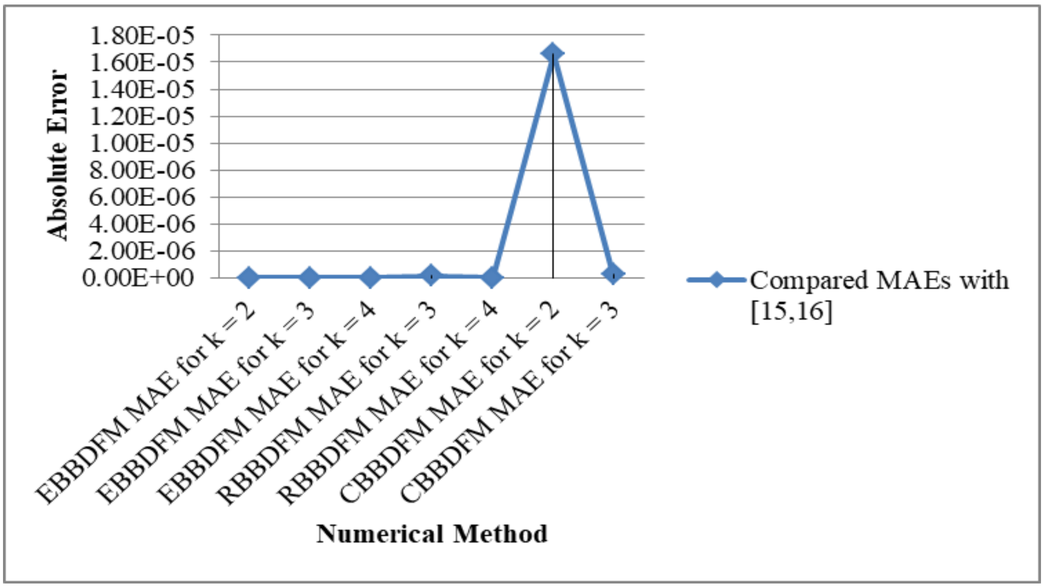

Figure 9: Compared Absolute Errors showing that the k – step number 4 performed better than the k – step numbers 3 and 2 by producing the Least Minimum Absolute Error.

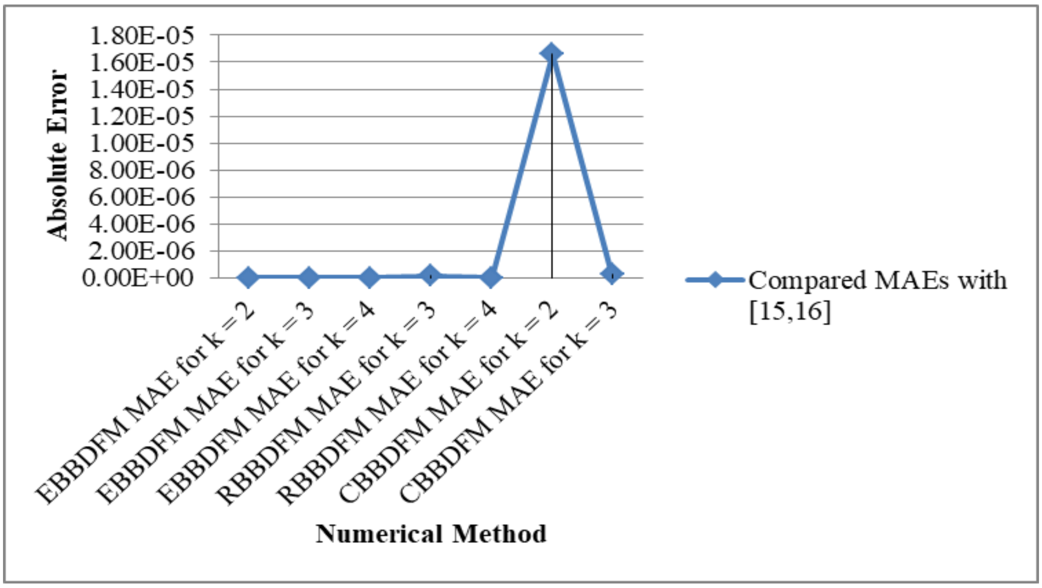

Figure 10: Compared Absolute Errors showing that the k – step number 4 performed better than the k – step numbers 3 and 2 by producing the Least Minimum Absolute Error.

5.4 Comparison of Results Base on the Computational Complexity

Under this sub-section, our focus is particularly on the Computational Time (CT) of the proposed method over other existing methods to prove its advantage. Computational Time (CT) is generally measured by the number of needed elementary operations carried-out using the step numbers k = 2, 3 and 4 discrete schemes of the proposed method to obtained the numerical solutions of the first order DDE and the memory storage requirements of the software and computer used. The Computational Time (CT) of this study is called Computational Processing Unit Time (CPUT). The step number of the suggested approach that generates the Least Minimum Absolute Error at Lower Computational Processing Unit Time (LCPUT) performs better than other step numbers in accuracy, efficiency, and faster computational time. This is because the LCPUT achieves the lowest possible absolute error.

Therefore, we compared the Computational Time (CT) of our method with other existing methods applied by other researchers in solving first order DDEs numerically to prove its superiority

Table 7: Comparison of Computational Time (CT) of EBBDFM $k = 2, 3$ and 4 with [15, 16] for constant step size z = 0.01 Using Problem 1

Numerical Method	k = 2 CT (s)	k = 3 CT (s)	k = 4 CT (s)
EBBDFM	4.00E-01	2.00E-01	1.00E-01
RBBDFM	6.00E-01	4.00E-01	3.00E-01
CBBDFM	7.00E-01	5.00E-01	4.00E-01

Table 8: Comparison of Computational Time (CT) of EBBDFM $k = 2,3$ and 4 with [15, 16] for constant step size z = 0.01 Using Problem 2

Numerical Method	k = 2 CT (s)	k = 3 CT (s)	k = 4 CT (s)
EBBDFM	5.00E-01	3.00E-01	2.00E-01
RBBDFM	8.00E-01	6.00E-01	5.00E-01
CBBDFM	9.00E-01	7.00E-01	6.00E-01

6. Conclusion

We found out if the discrete schemes shown in equations (2), (3), and (4) converge, are P-stable, and are Q-stable by looking at their corresponding continuous formulations. Tables 3, 4, 5, 6, 7, and 8, along with figures 7, 8, 9, and 10, which include numerical results and comparisons, showed that the EBBDFM scheme for step number 4 performed better than the EBBDFM schemes for steps 1 and 2 compared to other methods. The fact that it produced the least minimal absolute error at the smallest possible computational processing unit time (LCPUT) proved its accuracy and efficiency. These results were obtained using the newly developed mathematical expressions, as shown in tables 5, 6, 7, and 8, and in figures 7, 8, 9, and 10.

This is why we suggest that the EBBDFM schemes with more significant step numbers perform better than the EBBDFM schemes with lower step numbers. Thus, this study recommends that the new mathematical expressions developed for the evaluation of the delay term different from the existing formulas in literature performs better in producing accurate numerical results for first order DDE. In light of this, the step numbers of and for EBBDFM are appropriate for the solution of first-order differential equations. Additional research needs to be conducted to determine the step numbers for building discrete EBBDFM schemes for numerical solutions of first-order DDEs. This research should use the recently created formula for evaluating the delay term.

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